A NOTE ON THE R. FUCHS'S PROBLEM FOR THE PAINLEVÉ EQUATIONS

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ABSTRACT. In this article we consider a first-order completely integrable system of partial differential equations $\partial\Phi/\partial x = A(x,t)\,\Phi$, $\partial\Phi/\partial t = B(x,t)\,\Phi$ with $\Phi = (\phi_1, \phi_2)^\tau$ where $A(x,t)$ and $B(x,t)$ are 2×2 holomorphic matrix functions. Under some assumptions we find a variable change by which the system $\partial\Phi/\partial x = A(x,t)\,\Phi$ is reduced to an equation independent on the variable t. As an application we show that the R.Fuchs's conjecture for the Painlevé equations is true for some algebraic solutions.

Key words: Painlevé equations, Isomonodromic deformation, Ordinary differential equations

1. STATEMENT OF THE RESULT

In the first part of this note (see section 2) we consider a first-order linear system of partial differential equations

$$(1.1) \qquad \frac{\partial\Phi(x,t)}{\partial x} = A(x,t)\,\Phi(x,t)\,, \quad A(x,t) = \begin{pmatrix} a_{11}(x,t) & a_{12}(x,t) \\ a_{21}(x,t) & a_{21}(x,t) \end{pmatrix}$$

$$(1.2) \qquad \frac{\partial\Phi(x,t)}{\partial t} = B(x,t)\,\Phi(x,t)\,, \quad B(x,t) = \begin{pmatrix} b_{11}(x,t) & b_{12}(x,t) \\ b_{21}(x,t) & b_{21}(x,t) \end{pmatrix}$$

where $\Phi(x,t) = \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix}$ and $a_{ij}(x,t), b_{ij}(x,t)$ are holomorphic in a domain $\mathcal{D}$ in (x,t)-space.

We assume that:

Assumption A.1: The system (1.1) - (1.2) is completely integrable in the sense of the Frobenius theorem

Theorem 1.1. *(Frobenius) The system* (1.1) - (1.2) *is completely integrable if and only if*

$$(1.3) \qquad \frac{\partial A(x,t)}{\partial t} - \frac{\partial B(x,t)}{\partial x} + A(x,t)\,B(x,t) - B(x,t)\,A(x,t) = 0\,.$$

The system (1.3) is called the integrability condition of the system (1.1) - (1.2);

Assumption A.2: Both matrices $A(x,t)$ and $B(x,t)$ are traceless;

Assumption A.3: All components $a_{ij}(x,t)$ and $b_{ij}(x,t)$ of the matrices $A(x,t)$ and $B(x,t)$ respectively are of the form

$$\sum F_m(x)G_m(t)\,,$$

where $F_m(x)$ and $G_m(t)$ are holomorphic in the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ in $x-$ and $t-$ space respectively, such that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$.

We can transform the system (1.1) by a standard technique into a second order linear equation. It is easy to derive the following classical result:

Lemma 1.2. *Assume that $a_{12}(x,t)$ and $b_{12}(x,t)$ do not vanish identically. Then under Assumption A.1 and Assumption A.2 the first component $\phi_1(x,t)$ of the solution $\Phi(x,t)$ of the systems (1.1) - (1.2) satisfies the following differential equations*

$$(1.4) \qquad \frac{\partial^2 \phi_1(x,t)}{\partial x^2} + p_1(x,t)\frac{\partial \phi_1(x,t)}{\partial x} + q_1(x,t)\,\phi_1(x,t) = 0$$

$$(1.5) \qquad \frac{\partial \phi_1(x,t)}{\partial x} = p_2(x,t)\frac{\partial \phi_1(x,t)}{\partial t} + q_2(x,t)\,\phi_1(x,t)\,,$$

where

$$
(1.6) \qquad
\begin{aligned}
p_1(x,t) &= -\frac{\partial}{\partial x}\log a_{12}(x,t)\,,\\[2mm]
q_1(x,t) &= \det A(x,t) - \frac{\partial a_{11}(x,t)}{\partial x} + a_{11}(x,t)\frac{\partial}{\partial x}\log a_{12}(x,t)\,,\\[2mm]
p_2(x,t) &= \frac{a_{12}(x,t)}{b_{12}(x,t)}\,,\\[2mm]
q_2(x,t) &= a_{11}(x,t) - b_{11}(x,t)\frac{a_{12}(x,t)}{b_{12}(x,t)} =\\[2mm]
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\log b_{12}(x,t) - \frac{1}{b_{12}(x,t)}\frac{\partial a_{12}(x,t)}{\partial t}\right)\,.
\end{aligned}
$$

Let the elements $a_{12}(x,t)$ and $b_{12}(x,t)$ have the following simple forms

$$a_{12}(x,t) = g(t)\,[P_1(x) + t\,P_2(x)]\,,\quad b_{12}(x,t) = g(t)\,P_3(x)\,,$$

where $g(t)$ and $P_i(x)$ are holomorphic functions in $\mathcal{D}_2$ and $\mathcal{D}_1$ respectively. Then from the last equation of (1.6) we find that

$$(1.7) \qquad q_2(x,t) = R(x) + M(t)\,[f(x) + t\,h(x)]\,,$$

where

$$(1.8) \qquad f(x) = \frac{P_1(x)}{P_3(x)}\,,\quad h(x) = \frac{P_2(x)}{P_3(x)}$$

and $R(x)$ and $M(t)$ are functions dependent only on $P_i(x)$ and $g(t)$.

Our main result is the following

Theorem 1.3. *Assume that $a_{12}(x,t)$ and $b_{12}(x,t)$ do not vanish identically as*

$$(1.9) \qquad a_{12}(x,t) = g(t)\,[P_1(x) + t\,P_2(x)]\,,\quad b_{12}(x,t) = g(t)\,P_3(x)\,,$$

where $g(t)$ is a holomorphic function in $\mathcal{D}_2$, $P_i(x)$ are holomorphic functions in $\mathcal{D}_1$.

Then under Assumption A.i, $i = 1,2,3$ by means of the change the variables

$$(1.10) \qquad \phi_1(x,t) = \exp\left(\int R(x)dx\right)w(x,t)\,,$$

$$(1.11) \qquad \tau = t\exp\left(\int h(x)\,dx\right) + \int f(x)\exp\left(\int h(x)dx\right)dx$$

with $R(x), h(x)$ and $f(x)$ defined by (1.7), (1.8), equation (1.4) is reduced to the second order linear differential equation

(1.12)
$$\frac{d^2w}{d\tau^2} + P(\tau)\frac{dw}{d\tau} + Q(\tau)\,w = 0$$

which is independent on t.

Remark 1.4. As we are going to apply Theorem 1.3 to the R.Fuchs's principle for the Painlevé equation we formulate it from the point of view of the variable x. One can rewrite it and Lemma 1.2 from the point of view of the variable t. For example: equation (1.4) is considered as a second order equation

$$\frac{\partial^2 \phi_1(x,t)}{\partial t^2} - \frac{\partial}{\partial t}\log b_{21}(x,t)\frac{\partial \phi_1(x,t)}{\partial t} +$$

$$+ \left[\det B(x,t) + \frac{\partial b_{11}(x,t)}{\partial t} - b_{11}(x,t)\frac{\partial}{\partial t}\log b_{21}(x,t)\right]\phi_1(x,t) = 0$$

and this equation can be transformed by an appropriate variable change into an equation independent on the variable x.

The meaning of the transformations (1.10) - (1.11) is the following. Consider the auxiliary system to the quasilinear first-order partial differential equation (1.5)

$$\left|\begin{array}{rcl} \dot{x} &=& 1 \\ \dot{t} &=& -p_2(x,t) = -t\,h(x) - f(x) \qquad\qquad (*) \\ \dot{\phi}_1 &=& q_2(x,t)\,\phi_1 = (R(x) + M(t)\,[f(x) + t\,h(x)])\,\phi_1\,, \end{array}\right.$$

where $\dot{} = \frac{d}{ds}$. Then τ and $w(x,t)$ defined by

$$\tau = t\exp\left(\int h(x)\,dx\right) + \int f(x)\exp\left(\int h(x)dx\right)dx$$

$$w(x,t) = \phi_1 \exp\left(-\int R(x)\,dx\right)\exp\left(\int M(t)\,dt\right)$$

are two independent first integrals of the system $(*)$. As the exponent $\exp\left(\int M(t)\,dt\right)$ does not depend on the variable x we delete it from the transformation (1.10).

In the second part (see section 3) we relate Theorem 1.3 to the R.Fuchs's conjecture for the Painlevé equations. The six Painlevé equations govern the isomonodromic deformations of a linear system (1.1) with rational of x elements $a_{ij}(x,t)$ (the variable x is usually called the spectral parameter). The general theory of isomonodromic deformations ensures that the solution $\Phi(x,t)$ of (1.1) satisfies an additional linear system (1.2) (the variable t is called the deformation parameter). The integrability condition (1.3) of the systems (1.1) and (1.2) leads to the six Painlevé equations P_J, [2, 4, 9]. In such a case we refer to (1.1) - (1.2) as a linearization of the Painlevé equations. In [3] R.Fuchs made the following hypothesis: let $y(t)$ is an algebraic solution of the Painlevé equation, then there exists a suitable variable change by which the associated linear equation (1.1) can be transformed into an equation independent on the deformation parameter t. Moreover in the same paper [3] R.Fuchs showed that the linear equation taken at the solutions $y = 0, 1, \infty, t$ of P_{IV} (obtained as special Picard solutions) can be reduced to the hypergeometric equation. The R.Fuchs's conjecture for the Painlevé equations was recently reinvestigated in [7, 8].

Utilizing Theorem 1.3 we show that the R.Fuchs's hypothesis is true for some algebraic solutions of P_J.

This paper is organized as follows. In the next section we prove Theorem 1.3. In section 3 we apply Theorem 1.3 to the R.Fuchs's principle for the Painlevé equations from the second to the fifth.

2. Proof of the Theorem 1.3

We will prove the theorem in a general situation when all of the functions $M(t), f(x)$ and $h(x)$ are nonconstants. At the end of the proof we are going only to list equations (1.12) in the particular cases when some of these functions are constants.

Observe that the assumption of the theorem and the last equation of (1.6) imply

$$g(t) = B^{-1} \exp\left(-2 \int M(t)dt\right), \quad R(x) = \frac{1}{2}\left[\frac{P_3'(x)}{P_3(x)} - h(x)\right]$$

for a constant B.

Hence

$$a_{12}(x,t) = B^{-1}e^{-2\int M(t)dt}\left[P_1(x) + t\, P_2(x)\right], \quad b_{12}(x,t) = B^{-1}e^{-2\int M(t)dt}\, P_3(x).$$

On the other hand we have

$$a_{11}(x,t) - [f(x) + th(x)]\, b_{11}(x,t) = R(x) + M(t)\,[f(x) + th(x)]\,.$$

Then compatibility condition (1.3) of the linear system (1.1) - (1.2) gives

(2.13) $\dot{a}_{11} - b_{11}' + a_{12}b_{21} - b_{12}a_{21} = 0$

where $\dot{} := \frac{\partial}{\partial t}$, $' := \frac{\partial}{\partial x}$. In particular

$$a_{21}(x,t) - [f(x) + th(x)]\, b_{21}(x,t) =$$
$$= \frac{B\, e^{2\int M(t)dt}}{P_3(x)}\left[(M(t) + b_{11}(x,t))\, h(x) - b_{11}'(x,t) + (f(x) + th(x))\,(\dot{b}_{11}(x,t) + \dot{M}(t))\right].$$

This relation implies

$$a_{21}(x,t) = \frac{B\, e^{2\int M(t)dt}}{P_3(x)}\,\widetilde{a}_{21}(x,t), \quad b_{21}(x,t) = \frac{B\, e^{2\int M(t)dt}}{P_3(x)}\,\widetilde{b}_{21}(x,t)$$

as

$$\widetilde{a}_{21}(x,t) - [f(x) + th(x)]\,\widetilde{b}_{21}(x,t) =$$
$$= (M(t) + b_{11}(x,t))\, h(x) - b_{11}'(x,t) + (f(x) + th(x))\,(\dot{b}_{11}(x,t) + \dot{M}(t))\,.$$

Let us suppose that $b_{11}(x,t) = F_1(x)G_1(t) + F_2(x)G_2(t)$. Next, compatibility condition

(2.14) $\dot{a}_{21} - b_{21}' + 2a_{21}b_{11} - 2a_{11}b_{21} = 0$

gives equation

$$-\frac{\partial \widetilde{b}_{21}(x,t)}{\partial x} + [f(x) + th(x)]\,\frac{\partial \widetilde{b}_{21}(x,t)}{\partial t} = -2h(x)\,\widetilde{b}_{21}(x,t) -$$
$$-\; 2\,[M(t) + F_1(x)G_1(t) + F_2(x)G_2(t)] \times$$
$$\times\; \left[(f(x) + th(x))\left(\dot{M}(t) + F_1(x)\,\dot{G}_1(t) + F_2(x)\,\dot{G}_2(t)\right) +\right.$$
$$+\; h(x)\,(M(t) + F_1(x)G_1(t) + F_2(x)G_2(t)) - F_1'(x)G_1(t) - F_2'(x)G_2(t)\Big] -$$
$$-\; \frac{\partial}{\partial t}\Big[(f(x) + th(x))\left(\dot{M}(t) + F_1(x)\,\dot{G}_1(t) + F_2(x)\,\dot{G}_2(t)\right) +$$
$$+\; h(x)\,(M(t) + F_1(x)G_1(t) + F_2(x)G_2(t)) - F_1'(x)G_1(t) - F_2'(x)G_2(t)\Big].$$

We can write the general solution of this quasilinear first-order partial differential equation as

$$\begin{aligned}
\widetilde{b}_{21}(x,t) &= e^{2\int h(x)dx}\,F(\tau) - [F_1(x)G_1(t) + F_2(x)G_2(t) + M(t)]^2 -\\
&\quad - \frac{\partial}{\partial t}\,[F_1(x)G_1(t) + F_2(x)G_2(t) + M(t)] =\\
&= e^{2\int h(x)dx}\,F(\tau) - [b_{11}(x,t) + M(t)]^2 - \frac{\partial}{\partial t}\,[b_{11}(x,t) + M(t)]
\end{aligned}$$

for an arbitrary holomorphic function F of τ. Then the initial equation (1.4) is transformed to equation

(2.15)
$$\frac{d^2w}{d\tau^2} - F(\tau)\,w = 0\,,$$

which is independent on the variable t.

One can show, in a similar way, that when $b_{11}(x,t) = \sum F_i(x)\,G_i(t)$ equation (1.4) is again reduced to equation (2.15). This proves the theorem in a general situation.

We finish the proof with a list of particular situations. If $M(t) \equiv M$ for a nonzero constant M then:

- if both $f(x)$ and $h(x)$ do not vanish identically, no matter they are constants or not, then as above equation (1.4) gets into equation (2.15);
- if $f(x)$ does not vanish identically and $h(x) \equiv 0$ then equation (1.4) turns into equation

(2.16)
$$\frac{d^2w}{d\tau^2} - \left[M^2 + F(\tau)\right]\,w = 0$$

for an arbitrary holomorphic function F of $\tau = t + \int f(x)dx$;

- if $h(x)$ does not vanish identically and $f(x) \equiv 0$ then

$$a_{12}(x,t) = B^{-1}e^{-2Mt}\,t^{A+1}\,P_2(x), \quad b_{12}(x,t) = B^{-1}e^{-2Mt}\,t^A\,P_3(x)$$
$$R(x) = \frac{1}{2}\left[\frac{P_3'(x)}{P_3(x)} - (A+1)h(x)\right]$$

for constants B and A. Equation (1.4) is transformed to equation

(2.17)
$$\frac{d^2w}{d\tau^2} - \frac{A}{\tau}\frac{dw}{d\tau} - F(\tau)\,w = 0$$

for an arbitrary holomorphic function F of $\tau = t \exp\left(\int h(x)dx\right)$.

If $f(x)$ does not vanish identically then:

- if $M(t) \equiv 0, h(x) \equiv 0$ then

$$a_{12}(x,t) = B^{-1}e^{At}\,P_1(x)\,, \quad b_{12}(x,t) = B^{-1}e^{At}\,P_3(x)$$

$$R(x) = \frac{1}{2}\left[\frac{P_3'(x)}{P_3(x)} - Af(x)\right]$$

for constants B and A. Equation (1.4) turns into equation

(2.18)
$$\frac{d^2w}{d\tau^2} - A\frac{dw}{d\tau} - F(\tau)\,w = 0$$

for an arbitrary holomorphic function F of $\tau = t + \int f(x)dx$;

- We make note that the situation when both $f(x)$ and $h(x)$ do non vanish identically but $M(t) \equiv 0$ is impossible.

If $h(x)$ does not vanish identically then:

- if $M(t) \equiv 0, f(x) \equiv 0$ then

$$a_{12}(x,t) = B^{-1}t^{A+1}\,P_2(x)\,, \quad b_{12}(x,t) = B^{-1}t^A\,P_3(x)$$

$$R(x) = \frac{1}{2}\left[\frac{P_3'(x)}{P_3(x)} - (A+1)h(x)\right]$$

for constants B and A. Equation (1.4) is transformed to equation (2.17) for an arbitrary holomorphic function F of $\tau = t \exp\left(\int h(x)dx\right)$.

This proves the theorem. $\square$

We end this section in similar to Lemma 1.2 and Theorem 1.3 results about the second component $\phi_2(x,t)$ of the solution $\Phi(x,t)$ of the systems (1.1) - (1.2).

Lemma 2.1. *Assume that $a_{21}(x,t)$ and $b_{21}(x,t)$ do not vanish identically. Then under Assumption A.1 and Assumption A.2 the second component $\phi_2(x,t)$ of the solution $\Phi(x,t)$ of the systems (1.1) - (1.2) satisfies the following differential equations*

(2.19)
$$\frac{\partial^2\phi_2(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\log a_{21}(x,t)\frac{\partial\phi_2(x,t)}{\partial x} +$$

$$+ \left[\det A(x,t) + \frac{\partial a_{11}(x,t)}{\partial x} - a_{11}(x,t)\frac{\partial}{\partial x}\log a_{21}(x,t)\right]\phi_2(x,t) = 0$$

(2.20)
$$\frac{\partial\phi_2(x,t)}{\partial x} = \frac{a_{21}(x,t)}{b_{21}(x,t)}\frac{\partial\phi_2(x,t)}{\partial t} - \left[a_{11}(x,t) - b_{11}(x,t)\frac{a_{21}(x,t)}{b_{21}(x,t)}\right]\phi_2(x,t)\,.$$

Theorem 2.2. *Assume that $a_{21}(x,t)$ and $b_{21}(x,t)$ do not vanish identically as*

(2.21)
$$a_{21}(x,t) = g(t)\left[P_1(x) + t\,P_2(x)\right]\,, \quad b_{21}(x,t) = g(t)\,P_3(x)\,,$$

$$a_{11}(x,t) - b_{11}(x,t)\frac{a_{21}(x,t)}{b_{21}(x,t)} = R(x) + M(t)\left[f(x) + t\,h(x)\right]$$

where $M(t)$ is a function of t, $P_i(x)$ are functions of x and $h(x) = \frac{P_2(x)}{P_3(x)}$, $f(x) = \frac{P_1(x)}{P_3(x)}$.

Then under Assumption A.i, $i = 1, 2, 3$ by means of the change the variables

$$(2.22) \qquad \phi_2(x, t) = \exp\left(-\int R(x) dx\right) w(x, t),$$

$$(2.23) \qquad \tau = t \exp\left(\int h(x)\, dx\right) + \int f(x) \exp\left(\int h(x) dx\right) dx$$

equation (2.19) is reduced to the second order linear differential equation

$$(2.24) \qquad \frac{d^2 w}{d\tau^2} + P(\tau)\frac{dw}{d\tau} + Q(\tau)\, w = 0$$

which is independent on t.

3. The R. Fuchs's principle for the Painlevé equations

In this section applying Theorem 1.3 and Theorem 2.2, we show that the R.Fuchs's conjecture is true for some algebraic solutions of the Painlevé equations from the second to the fifth. Unfortunately the particular form of $q_2(x, t) = a_{11}(x, t) - b_{11}(x, t)\, a_{12}(x, t)/b_{12}(x, t)$ in (1.6) restricts our applications very much. On the other hand as the Assumption A.3 is not fulfilled for the sixth Painlevé equation we are not going to consider this Painlevé equation here. In fact we make no claim to try all possible applications of Theorem 1.3 and Theorem 2.2 in the R.Fuchs's principle for the Painlevé equations. We just give some examples.

3.1. The R. Fuchs's principle for P_{II}.

3.1.1. *Miwa - Jimbo's linearization.* Miwa - Jimbo's isomonodromic deformation equations for the second Painlevé equation [4] are

$$(3.25) \qquad \frac{\partial \Phi(x, t)}{\partial x} = A(x, t)\, \Phi(x, t),$$

$$A(x, t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x^2 + \begin{pmatrix} 0 & u \\ -2u^{-1}z & 0 \end{pmatrix} x +$$

$$+ \begin{pmatrix} z + t/2 & -uy \\ -2u^{-1}(\theta + yz) & -z - t/2 \end{pmatrix},$$

$$(3.26) \qquad \frac{\partial \Phi(x, t)}{\partial t} = B(x, t)\, \Phi(x, t),$$

$$B(x, t) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \frac{1}{2}\begin{pmatrix} 0 & u \\ -2u^{-1}z & 0 \end{pmatrix},$$

which $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ where y, z and u are functions of t and θ is a parameter.

Integrability condition (1.3) of the linear system (3.25) - (3.26) gives

$$\frac{dy}{dt} = z + y^2 + \frac{t}{2}, \qquad \frac{dz}{dt} = -2yz - \theta, \qquad \frac{d}{dt}\log u = -y.$$

Eliminating z, we obtain the second Painlevé equation P_{II}

$$\frac{d^2 y}{dt^2} = 2y^3 + ty + \alpha$$

with $\alpha = \frac{1}{2} - \theta$.

3.1.2. *R. Fuchs' conjecture for the solution $y \equiv 0$.* Equations (1.4) and (1.5) taken at the solution $y \equiv 0$, $\theta = 1/2\,(\alpha = 0)$ of P_{II} are

(3.27)
$$\frac{\partial^2 \phi_1}{\partial x^2} - \frac{1}{x}\frac{\partial \phi_1}{\partial x} - x^2\,(x^2 + t)\,\phi_1 = 0\,,$$
$$\frac{\partial \phi_1}{\partial x} = 2x\,\frac{\partial \phi_1}{\partial t}\,.$$

Under Theorem 1.3 we have:

$$M(t) \equiv 0, \;\; f(x) = 2x, \;\; h(x) \equiv 0, \;\; A = 0\,, \;\; R(x) \equiv 0\,.$$

By means of the change of the variable

$$x^2 + t = \tau$$

equation (3.27) is reduced to equation (2.18) taken at $A = 0$

$$\frac{d^2 \phi_1}{d\tau^2} = \frac{\tau}{4}\,\phi_1\,.$$

which is independent on the deformation parameter t. The last equation after transformation $\tau = 4^{1/3}\,\xi$ is converted into the Airy equation [1]

(3.28)
$$\frac{d^2 \phi_1}{d\xi^2} = \xi\,\phi_1\,.$$

We notice that when $\theta = 1/2$ we have $u = B^{-1}, z = -t/2$ for a constant B.

3.1.3. *R. Fuchs' conjecture for the solution $y = -\frac{1}{t}$.* Equations (2.19) and (2.20) taken at the solution $y = -\frac{1}{t}, \theta = -1/2\,(\alpha = 1)$ of P_{II} are are nothing but the equations (3.27)

$$\frac{\partial^2 \phi_2}{\partial x^2} - \frac{1}{x}\frac{\partial \phi_2}{\partial x} - x^2\,(x^2 + t)\,\phi_2 = 0\,,$$

$$\frac{\partial \phi_2}{\partial x} = 2x\,\frac{\partial \phi_2}{\partial t}\,.$$

which is reduced to the Airy equation (3.28), [1]

$$\frac{d^2 \phi_2}{d\xi^2} = \xi\,\phi_2\,.$$

We note that when $\theta = -1/2$ we have $u = B^{-1}t, z = -\frac{t}{2}$ for a constant B.

3.2. **The R. Fuchs's principle for P_{III}.**

3.2.1. *Miwa - Jimbo's linearization.* Miwa - Jimbo's isomonodromic deformation equations for the third Painlevé equation are [4]

$$(3.29) \qquad \frac{\partial \Phi(x,t)}{\partial x} = A(x,t)\,\Phi(x,t),$$

$$A(x,t) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} +$$

$$+ \frac{1}{x} \begin{pmatrix} -\theta_\infty/2 & -ywz \\ -w^{-1}\left((z-t)y + \frac{\theta_\infty+\theta_0}{2}\frac{z-t}{z} + \frac{\theta_\infty-\theta_0}{2}\right) & \theta_\infty/2 \end{pmatrix} +$$

$$+ \frac{1}{x^2} \begin{pmatrix} z - t/2 & -wz \\ w^{-1}(z-t) & -z+t/2 \end{pmatrix},$$

$$(3.30) \qquad \frac{\partial \Phi(x,t)}{\partial t} = B(x,t)\,\Phi(x,t),$$

$$B(x,t) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x +$$

$$+ \frac{1}{t} \begin{pmatrix} 0 & -ywz \\ -w^{-1}\left((z-t)y + \frac{\theta_\infty+\theta_0}{2}\frac{z-t}{z} + \frac{\theta_\infty-\theta_0}{2}\right) & 0 \end{pmatrix} +$$

$$+ \frac{1}{xt} \begin{pmatrix} -z+t/2 & wz \\ -w^{-1}(z-t) & z-t/2 \end{pmatrix},$$

with $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ where y, z and w are functions of t and θ_∞, θ_0 are parameters. Integrability condition (1.3) of the linear system (3.29) - (3.30) gives

$$t\frac{dy}{dt} = 4zy^2 - 2ty^2 + (2\theta_\infty - 1)\,y + 2t$$

$$t\frac{dz}{dt} = -4yz^2 + (4ty - 2\theta_\infty + 1)\,z + (\theta_0 + \theta_\infty)\,t,$$

$$t\frac{d}{dt}\log w = -\frac{(\theta_0 + \theta_\infty)t}{z} - 2ty + \theta_\infty.$$

Eliminating z we obtain the third Painlevé equation P_{III}

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}\left(\alpha\,y^2 + \beta\right) + \gamma\,y^3 + \frac{\delta}{y}$$

with

$$\alpha = 4\theta_0, \quad \beta = 4(1 - \theta_\infty), \quad \gamma = 4, \quad \delta = -4.$$

3.2.2. *R. Fuchs's principle for the solution $y \equiv 1$.* Equations (1.4) and (1.5) taken at the solution $y \equiv 1$, $\alpha + \beta = 0$ (resp. $\theta_0 = \theta_\infty - 1$) of P_{III} are

(3.31)
$$\frac{\partial^2 \phi_1}{\partial x^2} + \left[\frac{2}{x} - \frac{1}{x+1} \right] \frac{\partial \phi_1}{\partial x} +$$

$$+ \left[-\frac{t^2}{4} + \frac{1}{4(x+1)} + \frac{2t(\theta_\infty - 1) - 1}{4x} + \frac{8t^2 + 16(\theta_\infty - 1)t + 3}{16x^2} + \right.$$

$$\left. + \frac{(\theta_\infty - 1)t}{2x^3} - \frac{t^2}{4x^4} \right] \phi_1 = 0 \,,$$

$$\frac{\partial \phi_1}{\partial x} = \frac{t(x+1)}{x(x-1)} \frac{\partial \phi_1}{\partial t} + \left[\frac{\theta_\infty - 1}{2x} - \frac{2\theta_\infty - 1}{2(x-1)} - \frac{t(x+1)}{x(x-1)} \right] \phi_1 \,.$$

Accordingly Theorem 1.3 we have

$$M(t) \equiv -1 \,, \ f(x) \equiv 0 \,, \ h(x) = \frac{x+1}{x(x-1)} \,, \ A = \theta_\infty - 1 \,, \ R(x) = \frac{\theta_\infty - 1}{2x} - \frac{2\theta_\infty - 1}{2(x-1)} \,.$$

By means of the change of the variables

$$\phi_1(x,t) = x^{(\theta_\infty - 1)/2} \, (x-1)^{(1 - 2\theta_\infty)/2} \, w(x,t)$$

$$\tau = \frac{(x-1)^2 t}{x}$$

equation (3.31) is converted to equation (2.17)

$$\frac{d^2 w}{d\tau^2} - \frac{\theta_\infty - 1}{\tau} \frac{dw}{d\tau} + \left[-\frac{1}{4} + \frac{\theta_\infty - 1}{2\tau} + \frac{4\theta_\infty^2 - 1}{16\tau^2} \right] w = 0 \,,$$

which is independent on the deformation parameter t. Moreover after the change

$$w = \tau^{(\theta_\infty - 1)/2} \, v$$

the last differential equation is transformed to the Whittaker equation [10]

(3.32)
$$\frac{d^2 v}{d\tau^2} - \left(\frac{1}{4} - \frac{\kappa}{\tau} + \frac{4\mu^2 - 1}{4\tau^2} \right) v = 0$$

with parameters $\kappa = \frac{\theta_\infty - 1}{2}$, $\mu^2 = \frac{1}{16}$.

We make note that when $\theta_0 = \theta_\infty - 1$ we have $z = (1 - 2\theta_\infty)/4$, $w = B^{-1} e^{2t} t^{\theta_\infty}$ for a constant B.

3.3. **The R. Fuchs's principle for P_{IV}.**

3.3.1. *Miwa - Jimbo's linearization.* Miwa - Jimbo's isomonodromic deformation equations for the fourth Painlevé equation are [4]

$$(3.33) \qquad \frac{\partial \Phi(x,t)}{\partial x} = A(x,t)\,\Phi(x,t)\,,$$

$$A(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} t & u \\ 2(z-\theta_0-\theta_\infty)/u & -t \end{pmatrix} +$$

$$+ \frac{1}{x} \begin{pmatrix} -z+\theta_0 & -uy/2 \\ 2z(z-2\theta_0)/uy & z-\theta_0 \end{pmatrix},$$

$$(3.34) \qquad \frac{\partial \Phi(x,t)}{\partial t} = B(x,t)\,\Phi(x,t)\,,$$

$$B(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & u \\ 2(z-\theta_0-\theta_\infty)/u & 0 \end{pmatrix},$$

with $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ where u, z and y are functions of t and θ_0, θ_∞ are parameters. Integrability condition (1.3) of the linear system (3.33)-(3.34) gives

$$\frac{dy}{dt} = -4z + y^2 + 2t\,y + 4\theta_0$$

$$\frac{dz}{dt} = -\frac{2}{y}z^2 + \left(-y+\frac{4\theta_0}{y}\right)z + (\theta_0+\theta_\infty)\,y\,,$$

$$\frac{d}{dt}\log u = -y - 2t\,.$$

Eliminating z we obtain the fourth Painlevé equation P_{IV}

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4t\,y^2 + 2(t^2-\alpha)\,y + \frac{\beta}{y}$$

with

$$\alpha = 2\theta_\infty - 1\,, \quad \beta = -8\,\theta_0^2\,.$$

3.3.2. *R. Fuchs's principle for the solution $y = -2t$.* Equations (1.4) and (1.5) taken at the solutions $y = -2t$, $\theta_0 = \theta_\infty = \frac{1}{2}$ and $y = -2t$, $\theta_0 = -\theta_\infty = -\frac{1}{2}$ of P_{IV} are the same and they are

$$(3.35) \qquad \frac{\partial^2 \phi_1}{\partial x^2} + \frac{t}{x(x+t)}\frac{\partial \phi_1}{\partial x} - \left[\frac{1}{4x^2} + (x+t)^2 + \frac{1}{2x(x+t)}\right]\phi_1 = 0\,,$$

$$\frac{\partial \phi_1}{\partial x} = \left[1+\frac{t}{x}\right]\frac{\partial \phi_1}{\partial t} - \frac{1}{2x}\phi_1\,.$$

Under Theorem 1.3 we have

$$M(t) \equiv 0\,, \quad f(x) = 1\,, \quad h(x) = \frac{1}{x}\,, \quad A = 0\,, \quad R(x) = -\frac{1}{2x}\,.$$

By means of the change of the variables

$$\phi_1(x,t) = x^{-1/2}\,w(x,t)$$

$$\tau = t\,x + \frac{x^2}{2}$$

equation (3.35) is converted to equation (2.18) taken at $A = 0$

$$\text{(3.36)} \qquad \frac{d^2 w}{d\tau^2} - w = 0,$$

which is independent on the deformation parameter t.

We note that when $\theta_0 = -\theta_\infty = -\frac{1}{2}$ and $\theta_0 = \theta_\infty = \frac{1}{2}$ we have $u = B^{-1}$ and $z = 0, z = 1$ respectively for a constant B.

3.3.3. *R. Fuchs's principle for the solution $y = -\frac{2}{3} t$.* Equations (1.4) and (1.5) taken at the solutions $y = -\frac{2}{3} t$, $\theta_\infty = \frac{1}{2}$, $\theta_0 = -\frac{1}{6}$ and $y = -\frac{2}{3} t$, $\theta_\infty = \frac{1}{2}$, $\theta_0 = \frac{1}{6}$ of P_{IV} are the same and they are

$$\text{(3.37)} \quad \frac{\partial^2 \phi_1}{\partial x^2} + \frac{t}{x(3x+t)} \frac{\partial \phi_1}{\partial x} - \left[\frac{7}{36x^2} - \frac{t}{6x^2(3x+t)} + \frac{(3x+t)^2 (3x+4t)}{27x} \right] \phi_1 = 0,$$

$$\frac{\partial \phi_1}{\partial x} = \left[1 + \frac{t}{3x} \right] \frac{\partial \phi_1}{\partial t} + \left[-\frac{1}{6x} + \frac{2t}{3} \left(1 + \frac{t}{3x} \right) \right] \phi_1.$$

Under Theorem 1.3 we have

$$M(t) = \frac{2t}{3}, \quad f(x) = 1, \quad h(x) = \frac{1}{3x}, \quad R(x) = -\frac{1}{6x}.$$

By the transformation

$$\phi_1 = x^{-1/6} w, \quad \tau = tx^{1/3} + \frac{3}{4} x^{4/3}$$

equation (3.37) is reduced to equation (2.15)

$$\frac{d^2 w}{d\tau^2} = \frac{4\tau}{3} w,$$

which is independent on deformation parameter t. After the change $\tau = \left(\frac{3}{4} \right)^{1/3} \xi$ the last equation is converted into the Airy equation (3.28), [1]

$$\frac{d^2 w}{d\xi^2} = \xi w.$$

We make note that when $\theta_\infty = \frac{1}{2}, \theta_0 = -\frac{1}{6}$ and $\theta_\infty = \frac{1}{2}, \theta_0 = \frac{1}{6}$ we have $u = B^{-1} e^{-\frac{2t^2}{3}}$ and $z = -\frac{2}{9} t^2, z = -\frac{2}{9} t^2 + \frac{1}{3}$ respectively for a constant B.

3.4. The R. Fuchs's principle for P_V.

3.4.1. *Miwa - Jimbo's linearization.* Miwa - Jimbo's isomonodromic deformation equations for the fifth Painlevé equation are [4]

$$\text{(3.38)} \qquad \frac{\partial \Phi(x,t)}{\partial x} = A(x,t) \, \Phi(x,t),$$

$$A(x,t) = \frac{1}{2} \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} + \frac{1}{x} \begin{pmatrix} z + \frac{\theta_0}{2} & -u(z + \theta_0) \\ u^{-1} z & -z - \frac{\theta_0}{2} \end{pmatrix} +$$

$$+ \frac{1}{x-1} \begin{pmatrix} -z - \frac{\theta_0 + \theta_\infty}{2} & uy \left(z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) \\ -\frac{1}{uy} \left(z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) & z + \frac{\theta_0 + \theta_\infty}{2} \end{pmatrix},$$

(3.39) $\dfrac{\partial \Phi(x,t)}{\partial t} = B(x,t)\,\Phi(x,t)\,,$

$$B(x,t) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x +$$

$$+ \frac{1}{t}\begin{pmatrix} 0 & -u\left[z + \theta_0 - y\left(z + \frac{\theta_0-\theta_1+\theta_\infty}{2}\right)\right] \\ u^{-1}\left[z - \frac{1}{y}\left(z + \frac{\theta_0+\theta_1+\theta_\infty}{2}\right)\right] & 0 \end{pmatrix},$$

with $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ where y, z and u are functions of t and θ_0, θ_1 and θ_∞ are parameters.

Integrability condition (1.3) of the linear system (3.38)-(3.39) gives

$$t\frac{dy}{dt} = ty - 2z(y-1)^2 - (y-1)\left(\frac{\theta_0-\theta_1+\theta_\infty}{2}y - \frac{3\theta_0+\theta_1+\theta_\infty}{2}\right),$$

$$t\frac{dz}{dt} = yz\left(z + \frac{\theta_0-\theta_1+\theta_\infty}{2}\right) - \frac{z+\theta_0}{y}\left(z + \frac{\theta_0+\theta_1+\theta_\infty}{2}\right),$$

$$t\frac{d}{dt}\log u = -2z - \theta_0 + y\left(z + \frac{\theta_0-\theta_1+\theta_\infty}{2}\right) + \frac{1}{y}\left(z + \frac{\theta_0+\theta_1+\theta_\infty}{2}\right).$$

Eliminating z we obtain the fifth Painlevé equation P_V

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}$$

with

$$\alpha = \frac{1}{2}\left(\frac{\theta_0-\theta_1+\theta_\infty}{2}\right)^2,\quad \beta = -\frac{1}{2}\left(\frac{\theta_0-\theta_1-\theta_\infty}{2}\right)^2,\quad \gamma = 1-\theta_0-\theta_1,\quad \delta = -\frac{1}{2}.$$

3.4.2. *R. Fuchs's principle for the solution* $y = 1 - \frac{t}{\theta_1-1}$. Equations (1.4) and (1.5) taken at the solution $y = 1 - \frac{t}{\theta_1-1}, \theta_0 = 0, \theta_1 + \theta_\infty = 2$ of P_V are

(3.40) $\dfrac{\partial^2\phi_1}{\partial x^2} + \dfrac{1}{x-1}\dfrac{\partial\phi_1}{\partial x} - \left[\dfrac{t^2}{4} + \dfrac{t(\theta_1-1)}{2(x-1)} + \dfrac{\theta_1^2}{4(x-1)^2}\right]\phi_1 = 0\,,$

$$\frac{\partial\phi_1}{\partial x} = \frac{t}{x-1}\frac{\partial\phi_1}{\partial t} - \left[\frac{2-\theta_1}{2(x-1)} + \frac{t}{2(x-1)}\right]\phi_1\,.$$

Accordingly Theorem 1.3 we have

$$M(t) = -\frac{1}{2},\quad f(x) \equiv 0,\quad h(x) = \frac{1}{x-1},\quad A = 1-\theta_1,\quad R(x) = \frac{\theta_1-2}{2(x-1)}\,.$$

By means of the change of the variables

$$\phi_1(x,t) = (x-1)^{(\theta_1-2)/2}\,w(x,t)\,,$$
$$\tau = t(x-1)$$

equation (3.40) is converted to equation (2.17)

$$\frac{d^2w}{d\tau^2} - \frac{1-\theta_1}{\tau}\frac{dw}{d\tau} + \left[-\frac{1}{4} + \frac{1-\theta_1}{2\tau} + \frac{1-\theta_1}{\tau^2}\right]w = 0\,,$$

which is independent on the deformation parameter t. Moreover, we can apply the transformation

$$w = \tau^{(1-\theta_1)/2} \, v$$

to the last differential equation and reduce it to the Whittaker equation (3.32), [10], with parameters $\kappa = (1 - \theta_1)/2$, $\mu^2 = \theta_1^2/4$.

We make note that when $\theta_0 = 0$, $\theta_1 + \theta_\infty = 2$, $y = 1 - \frac{t}{\theta_1 - 1}$ we have $u = \frac{B^{-1} t^{2-\theta_1} e^{-t}}{\theta_1 - 1 - t}$ and $z \equiv 0$ for a constant B.

3.4.3. *R. Fuchs's principle for the solution $y \equiv -1$.* In [5] Kazuo Kaneko and Yousuke Ohyama show that R. Fuchs's conjecture is true for the rational solution $y \equiv -1$ for $\theta_0 = \theta_1 = 1/2$ and arbitrary θ_∞ of the fifth Painlevé equation. We shall give this example as an application of Theorem 1.3. We remark that our transformations are slightly different from these in [5].

Equations (1.4) and (1.5) taken at the solution $y \equiv -1$ for $\theta_0 = \theta_1 = 1/2$ and arbitrary θ_∞ (see also [5]) of P_V are

$$(3.41) \qquad \frac{\partial^2 \phi_1}{\partial x^2} + \left[\frac{1}{x} + \frac{1}{x-1} - \frac{2t}{2(1-\theta_\infty) + 2xt - t} \right] \frac{\partial \phi_1}{\partial x} +$$

$$+ \left[-\frac{t^2}{4} - \frac{1}{16x^2} - \frac{1}{16(x-1)^2} + \frac{4(\theta_\infty - 1)t^2}{t^2 - 4(1-\theta_\infty)} \cdot \frac{1}{2(1-\theta_\infty) + 2tx - t} + \right.$$

$$+ \left(\frac{t^2}{16} + \frac{\theta_\infty t}{4} - \frac{t}{4} + \frac{1 - \theta_\infty}{t - 2(1-\theta_\infty)} + \frac{2\theta_\infty^2 - 4\theta_\infty + 3}{8} \right) \frac{1}{x} +$$

$$+ \left. \left(-\frac{t^2}{16} + \frac{\theta_\infty t}{4} - \frac{t}{4} + \frac{1 - \theta_\infty}{t + 2(1-\theta_\infty)} - \frac{2\theta_\infty^2 - 4\theta_\infty + 3}{8} \right) \frac{1}{x-1} \right] \phi_1 \,,$$

$$\frac{\partial \phi_1}{\partial x} = \left[\frac{1 - \theta_\infty}{x(x-1)} + \frac{t}{2} \cdot \frac{2x-1}{x(x-1)} \right] \frac{\partial \phi_1}{\partial t} +$$

$$+ \left[-\frac{2x-1}{4x(x-1)} - \frac{1}{4} \left(\frac{1 - \theta_\infty}{x(x-1)} + \frac{t}{2} \cdot \frac{2x-1}{x(x-1)} \right) \right] \phi_1 \,.$$

Under Theorem 1.3 we have

$$M(t) \equiv -\frac{1}{4}, \quad f(x) = \frac{1 - \theta_\infty}{x(x-1)}, \quad h(x) = \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x-1} \right), \quad R(x) = -\frac{1}{4x} - \frac{1}{4(x-1)} \,.$$

By means of the change of the variables

$$\phi_1(x,t) = (x(x-1))^{-1/4} \, w(x,t) \,,$$

$$\tau = t \, (x(x-1))^{1/2} - (1 - \theta_\infty) \log \frac{\sqrt{x} - \sqrt{x-1}}{\sqrt{x} + \sqrt{x-1}}$$

equation (3.41) is converted to equation (2.15)

$$\frac{d^2 w}{d\tau^2} - \frac{1}{4} \, w = 0$$

which is independent on the deformation parameter t.

We note that when $\theta_0 = \theta_1 = 1/2$ and $y \equiv -1$ we have $z = -(t + 2 + 2\theta_\infty)/8$ and $u = B^{-1} e^{t/2}$ for a constant B.

3.4.4. *Kitaev's linearization.* Kitaev's isomonodromic deformation equations for the degenerate fifth Painlevé equation with $\delta = 0$ are [6]

$$
(3.42) \qquad \frac{\partial \Phi(x,t)}{\partial x} = A(x,t)\,\Phi(x,t),
$$

$$
A(x,t) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} + \frac{1}{x-1} \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix},
$$

$$
(3.43) \qquad \frac{\partial \Phi(x,t)}{\partial t} = B(x,t)\,\Phi(x,t),
$$

$$
B(x,t) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & -(a_1 + b_1) \end{pmatrix},
$$

where a_i, b_i are functions of t. Let $y = y(t)$ is an arbitrary solution of P_V with parameters

$$
\alpha = \frac{\theta_0^2}{8}, \quad \beta = -\frac{\theta_1^2}{8}, \quad \gamma = \theta_\infty, \quad \delta = 0.
$$

Then a_i, b_i are defined as follows

$$
a_2 = \frac{\theta_\infty}{2(y-1)}, \quad \frac{d}{dt}\left(t\frac{d}{dt}\log a_2\right) = \theta_\infty \frac{d}{dt}\left(\frac{a_1}{a_2}\right) + 2a_2 + \theta_\infty,
$$

$$
a_3 = \frac{1}{a_2}\left(-\frac{\theta_0^2}{16} - a_1^2\right), \quad b_1 = \frac{t}{2}\frac{d}{dt}\log a_2 - a_1\left(1 + \frac{\theta_\infty}{2a_2}\right),
$$

$$
b_2 = -a_2 - \frac{\theta_\infty}{2}, \quad b_3 = -\frac{t}{a_2}\frac{d}{dt}a_1 - a_3\left(1 + \frac{\theta_\infty}{2a_2}\right).
$$

3.4.5. *R. Fuchs's principle for the solution $y = 1 + \kappa\sqrt{t}$.* In this subsection we show that R. Fuchs's principle is true for the algebraic solution $y = 1 + \kappa\sqrt{t}$, $\alpha = \mu$, $\beta = -1/8$, $\gamma = -\mu\kappa^2$, $\delta = 0$ for arbitrary constants κ and μ of P_V.

To apply Theorem 1.3 we make the transformation

$$
t \longmapsto z^2.
$$

Then equations (1.4) and (1.5) taken at the solution $y = 1 + \kappa z$ are

$$
(3.44) \qquad \frac{\partial^2 \phi_1}{\partial x^2} + \left[\frac{1}{x} + \frac{1}{x-1} - \frac{\kappa z}{\kappa z x + 1}\right]\frac{\partial \phi_1}{\partial x} +
$$

$$
+ \left[\frac{\mu}{2x^2} - \frac{1}{16(x-1)^2} + \frac{4\mu\kappa z + 2\mu - 1}{4x} + \frac{\kappa^2 z^2}{4(\kappa z + 1)(1 + \kappa z x)} - \right.
$$

$$
\left. - \frac{2\mu\kappa^3 z^3 + 6\mu\kappa^2 z^2 + 6\mu\kappa z + 2\mu - 1}{4(\kappa z + 1)(x-1)}\right]\phi_1 = 0
$$

$$
\frac{\partial \phi_1}{\partial x} = \left[\frac{1}{2\kappa x(x-1)} + \frac{z}{2(x-1)}\right]\frac{\partial \phi_1}{\partial z} +
$$

$$
+ \left[-\frac{1}{4(x-1)} + \frac{1}{2z}\left(\frac{1}{2\kappa x(x-1)} + \frac{z}{2(x-1)}\right)\right]\phi_1.
$$

Accordingly Theorem 1.3 we have

$$M(s) = \frac{1}{2z}, \quad f(x) = \frac{1}{2\kappa x(x-1)}, \quad h(x) = \frac{1}{2(x-1)}, \quad R(x) = -\frac{1}{4(x-1)}.$$

By means of the change of the variables

$$\phi_1(x,z) = (x-1)^{-1/4} w(x,z),$$

$$\tau = z(x-1)^{1/2} - \frac{i}{2\kappa} \log \frac{\sqrt{x-1}-i}{\sqrt{x-1}+i}$$

equation (3.44) is converted to equation (2.15)

$$\frac{d^2 w}{d\tau^2} - 2\mu\kappa^2 \, w = 0$$

which is independent on the deformation parameter t.

We make note that when $y = 1 + \kappa z$, $\alpha = \mu$, $\beta = -1/8$, $\gamma = -\mu\kappa^2$, $\delta = 0$ we have $a_1 = \left(-\frac{2z}{\kappa} - z^2 + B\right) a_2$ for a constant B.

4. Concluding Remarks

We shall address the generalization of Theorem 1.3 and new applications of this theorem in the forthcoming papers.

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